Tangent Planes

As mentioned in the previous worksheet, our goal is to do something similar to finding the equation of a tangent line to a surface. Only now we have two options for slope. They are given by partial derivatives. We'll put these things together to find equations of tangents planes to surfaces. There will be two cases: when the surface is given by the graph of a function, and when the surface is given by a parameterization. We'll start with the second of these two and see that the former is just a special case of the latter.

1. In the following we will find the tangent plane to the surface S given by $\vec{r}(u,v) = \langle u\cos(v), 2u\sin(v), 10-e^{5u^2}\rangle$ with $(u,v) \in [0,\infty)\times[0,2\pi)$ at the point $(-1,2,10-e^{10})$.

   (a) For what values of (u,v) is $\vec{r}(u,v) = \langle -1, 2, 10-e^{10}\rangle$?

   (b) Consider the parametric curve contained in S given by $\vec{r}(u, 3\pi/4)$. Find the tangent vector to this curve at $u = \sqrt{2}$.

   (c) Now consider the curve in S given by $\vec{r}(\sqrt{2}, v)$. Find the tangent vector to this curve at $v = 3\pi/4$.

(d) Since the previous two curves are contained in the surface S then what must be true about the two vectors you found above?

(e) Using your two vectors, write down a *parametric* equation of the tangent plane to the surface S at the point given at the beginning of this problem.

(f) Now write down the equation of a tangent plane to S at the same point in the usual way (i.e. $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$).

So now we've seen that for a parametric surface $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ that two natural tangent vectors at a point $\vec{r}(u_0, v_0)$ are given by

$$\vec{r}_u(u_0, v_0) = \frac{\partial x}{\partial u}(u_0, v_0)\vec{i} + \frac{\partial y}{\partial u}(u_0, v_0)\vec{j} + \frac{\partial z}{\partial u}(u_0, v_0)\vec{k}$$

$$\vec{r}_v(u_0, v_0) = \frac{\partial x}{\partial v}(u_0, v_0)\vec{i} + \frac{\partial y}{\partial v}(u_0, v_0)\vec{j} + \frac{\partial z}{\partial v}(u_0, v_0)\vec{k}$$

2. Now we'll think about the case of a surface given by a graph. To do this we recall that a surface given by a graph $z = f(x, y)$ can be written as a parameterized surface given by $\vec{r}(x, y) = \langle x, y, f(x, y)\rangle$.

   (a) Write down both tangent vectors to $\vec{r}(x, y) = \langle x, y, f(x, y)\rangle$ at some point $(x_0, y_0, f(x_0, y_0))$.

   (b) write down the equation of a tangent plane at the point $(x_0, y_0, f(x_0, y_0))$.

   (c) Write the parameterized surface from exercise 1 as a graph $z = f(x, y)$. Hint: what do you notice about $x(u, v)^2 + y(u, v)^2/4$ and $z(u, v)$?

   (d) Now use your general equation from part (b) of this exercise to write down an equation of the tangent plane to the graph you found in part (c) at the point from exercise 1. Does this agree with your equation of a tangent plane from exercise 1?